# Appendix 4B THE TOURINHO MODEL & INCENTIVES

Many regard Tourinho (1979a)[1] as the father of real option methodology and application. The Ph.D. dissertation of Octavio Tourinho (1979b) (supervised by Hayne Leland) at the University of California (Berkeley) considers an unexploited petroleum reserve as a call option.

Tourinho first establishes that under either certainty or uncertainty, the reserve should never be extracted, which is an extraction paradox. If the expected asset yield in the Samuelson model is near or at zero, then $\beta_1=1$, and V* approaches infinity. However, since we observe prospectors exercising their call options and extracting the reserve within a finite time, incentives are required to induce the holders to exercise the option. Tourinho solves the extraction paradox by introducing a holding cost per unit of time that the prospector has to pay the option seller prior to the exercise event, and derives a closed-form solution for the option value. Also this approach provides a menu of policy options for a real option writer to influence the investment timing or optimal exercise of the real option.

1. Option Holding Cost

There is a sizeable body of real option analyses on decisions affecting the exploration and extraction industries. Governments and other landowners normally hold title to any subterranean reserves existing within their land or country boundaries and lease eligible land tracts (perhaps covered by water) for the purposes of exploring and extracting the natural resource. The lease purchase price is the amount paid by the leaseholder for the exploration and extraction rights. It is evaluated as the value of a deferral option and is determined per unit of reserve by the probability of discovering reserves, by the underlying asset price and its properties, the investment and operating costs, royalty and bonus payments, and the lease rentals. These rentals are the option holding charges that are paid to the owner (government) until the leaseholder starts extracting the reserve. The size of these rentals is therefore crucial to understanding the lease payment structure incurred by the leaseholder as well as the extent of any government control over leaseholder behaviour. However, the role of the rentals in determining the lease value is almost entirely neglected by the real option analytical literature

The concept of an option holding cost extends beyond the confines of the exploration and extraction industries. There are potential applications of a similar nature in the field of property development. Developers can acquire vacant plots or abandoned sites, with planning permission that embed the right to construct a property, at a price that is evaluated as a deferral option. During the construction phase, or in the absence of any construction activities, the government can impose a continuous charge (or land tax) on the developer, which remains in force until the construction is completed and sold. This charge represents an option holding cost and acts as a deterrent to the developer intent on postponing the property development. Although these illustrations depict the option holding cost as being imposed by the government, it is conceivable to also consider the option holding cost as a natural element of the firm's cost structure. The launch of a new product development may be deferred while inadequacies in complementary assets are rectified. But during this period of postponement, the firm incurs a continuous option holding cost since it has to sustain the product advantages to ensure that it always outperforms any potential rivals.

The critical shortcomings of the Tourinho model with an option holding cost lie in the restriction it places on the solution and, possibly, the absence of the convenience yield for the extracted reserve. But an investment opportunity model that includes both the convenience yield and an option holding cost produces a solution without restrictions, which lies somewhere between the real option and NPV solutions. This means that the option holding cost acts as a mediator between the real option and NPV solutions. Also, if the investment opportunity is a lease on a reserve and the option holding cost value is controlled by the government, then it can exert a degree of influence over the start of the resource extraction process as well as over the holder's profit level at exercise. Since a positive option holding cost lowers the lease option price, this structure is also attractive to the lessee because of the reduced upfront lease purchase price. Despite having a lower option price, a government as the lessor might use the modified Tourinho model to select a combination of holding costs, eventual royalities and compensation for writing a positive initial option which maximises the total value generated by any leasing deal.

## 2. Extraction Paradox and its Resolution

The Tourinho model is formulated in terms of one unit of extracted resource, V denotes the stochastic resource price, while the per unit extraction cost $X$ is the option exercise price. This cost is interpreted as the sum of the unit cost of mining, refining and transporting and the per unit rental cost of capital equipment. The resource price behaves according to:

$$dV = \mu V dt + \sigma V dz, \tag{1}$$

where $\mu$ denotes the required constant return for the underlying asset, and $\sigma$ its volatility. By contingent claims analysis, the risk neutral valuation relationship for the call option value $C$ is:

$$\tfrac{1}{2}\sigma^2 V^2 \frac{\partial^2 C}{\partial V^2} + rV\frac{\partial C}{\partial V} - rC = 0. \tag{2}$$

The general solution to (2) is supplied by Samuelson (1965):

$$C(V) = A_{11}V^{\beta_{11}} + A_{12}V^{\beta_{12}}, \tag{3}$$

with characteristic roots $\beta_{11} = 1, \beta_{12} = -2r/\sigma^2$. Since the option value tends to zero as the asset price approaches zero, $A_{12} = 0$. The exercise boundary for the asset price $\hat{V}$ is defined by the value matching relationship where the net revenue $\hat{V} - X$ is just sufficient to compensate for the option value $C(\hat{V})$. By imposing the smooth pasting condition, it is straightforward to show that:

$$\hat{V} = \frac{\beta_1}{\beta_1 - 1} X. \tag{4}$$

Since $\beta_1 = 1$, no finite price $\hat{V}$ exists and the paradox is demonstrated.

Tourinho (1979a) overcomes the paradox by introducing an option holding cost. While holding the extraction option, the prospector pays a constant cost per unit of time until the option is exercised, which represents a regular fee for keeping the option open that is paid to the option seller, who holds title to the geographic area containing the reserve. The effect of an accumulating holding cost is to motivate early exercise.

The positive holding cost per unit of time is denoted by $h$. Its introduction modifies the valuation relationship (2) to:

$$\tfrac{1}{2}\sigma^2 V^2 \frac{\partial^2 C}{\partial V^2} + rV\frac{\partial C}{\partial V} - rC = h. \tag{5}$$

The general solution to (5) takes the form:

$$C(V) = A_{21}V^{\beta_{11}} + A_{22}V^{\beta_{12}} - \frac{h}{r}. \tag{6}$$

Given (6), it is conceivable that the call option value $C$ is negative for some feasible V value. This possibility is restricted by formulating an inferior level $Z$ such that the option has a zero value for all asset prices below this level, that is $C(V) = 0$ for $V \le Z$. Secondly, there is some superior level $\hat{V}$ at which the option holder exercises the extraction option when the net revenue from extraction is just sufficient to reimburse the option value. The valuation function $C(V)$ can be expressed as:

$$C(V) = \begin{cases} 0 \text{ for } V \le Z, \\ A_{21}V + A_{22}V^{\beta_{12}} - \dfrac{h}{r} \text{ for } Z < V < \hat{V}, \\ V - X \text{ for } V \ge \hat{V}, \end{cases} \tag{7}$$

where $\beta_{11} = 1$. The optimal option policy depends on the prevailing spot price. If $V \le Z$, then the option having zero value is allowed to lapse, while if $V \ge \hat{V}$, then extraction is warranted since the net revenue from extracting the reserve exceeds the option value. For $Z < V < \hat{V}$, the reserve is more valuable in an unexploited state and a prospector would be willing to pay the upfront option premium $C(V)$ together with the regular holding cost h in order to secure the extraction rights. Tourinho (1979a) determines explicit closed-form solutions for $\hat{V}$ and $Z$ from the two value matching relationships and associated smooth pasting conditions:

$$\hat{V} = \frac{\beta_{12}}{\beta_{12}-1}\left(X - \frac{h}{r}\right)\left[1 - \left(1 - \frac{rX}{h}\right)^{-\left(\frac{\beta_{12}-1}{\beta_{12}}\right)}\right], \tag{8}$$

$$Z = \frac{\beta_{12}}{\beta_{12}-1}\frac{h}{r}\left[1 - \left(1 - \frac{rX}{h}\right)^{\left(\frac{\beta_{12}-1}{\beta_{12}}\right)}\right]. \tag{9}$$

Feasible solution values for $\hat{V}$ and $Z$ only exist provided h>rX. If h=rX, then $Z > \hat{V} = 0$ which contradicts the model formulation, and if h<rX, then both $\hat{V}$ and $Z$ are undefined.

The option value is a convex function and for $V > Z$, the option becomes increasingly more valuable as the spot price rises. This effect is illustrated in Figure 1B, which displays the option value for the base case values in the Figure. The profile is similar to the option value for the one-factor deferral investment model of Samuelson (1965) except for a zero option value when the spot price V dips below the inferior level $Z$.

Tourinho does establish analytically that a holding cost increase raises the inferior level but lowers the superior level and that a volatility increase lowers the inferior level but raises the superior level. This implies that positive changes in the holding cost and volatility lead, respectively, to decreases and increases in the option value.

Figure 1B

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | TOURINHO 79 | | | | | | |
| 2 | | | | | INPUT | | | | | | |
| 3 | V | 18 | 18.5 | 19 | 19.5 | 20 | 20.5 | 21 | 21.5 | 22 | 22.5 |
| 4 | r | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| 5 | σ | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 |
| 6 | $\sigma^2$ | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 |
| 7 | H | 3.5 | 3.5 | 3.5 | 3.5 | 3.5 | 3.5 | 3.5 | 3.5 | 3.5 | 3.5 |
| 8 | X | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |
| 9 | **OUTPUT** | | | | | | | | | | |
| 10 | $\beta_{12}$ | -5.00 | -5.00 | -5.00 | -5.00 | -5.00 | -5.00 | -5.00 | -5.00 | -5.00 | -5.00 |
| 11 | A1 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 | 1.57 |
| 12 | A2 | 13,039,176 | 13,039,176 | 13,039,176 | 13,039,176 | 13,039,176 | 13,039,176 | 13,039,176 | 13,039,176 | 13,039,176 | 13,039,176 |
| 13 | V* | 22.05 | 22.05 | 22.05 | 22.05 | 22.05 | 22.05 | 22.05 | 22.05 | 22.05 | 22.05 |
| 14 | Z | 18.62 | 18.62 | 18.62 | 18.62 | 18.62 | 18.62 | 18.62 | 18.62 | 18.62 | 18.62 |
| 15 | C(V) | 0.000 | 0.000 | 0.036 | 0.178 | 0.411 | 0.721 | 1.096 | 1.525 | 2.000 | 2.500 |
| 16 | MAX(V-X,0) | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.500 | 1.000 | 1.500 | 2.000 | 2.500 |
| 17 | ROV | | | 0.036 | 0.178 | 0.411 | 0.721 | 1.096 | 1.525 | 2.000 | |
| 18 | $\beta_{12}$ | -2*B4/B6 | | | | | | | | | |
| 19 | A1 | (1-(1-B4*B8/B7)^((B10-1)/B10))^-1 | | | | | | | | | |
| 20 | A2 | -B11/(B10*B14^(B10-1)) | | | | | | | | | |
| 21 | V* | (B10/(B10-1))*(B8-B7/B4)*(1/(1-B11)) | | | | | | | | | |
| 22 | Z | (B10/(B10-1))*(B7/B4)*(1/B11) | | | | | | | | | |
| 23 | ROV | B11*B3+B12*B3^B10-B7/B4 | | | | | | | | | |
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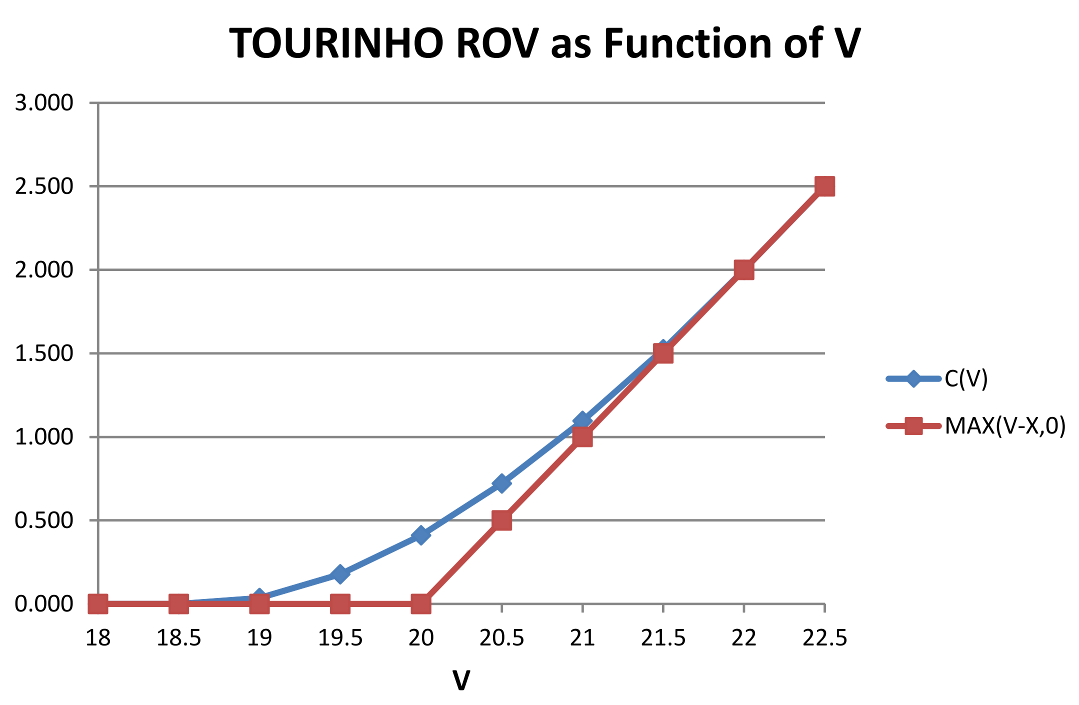
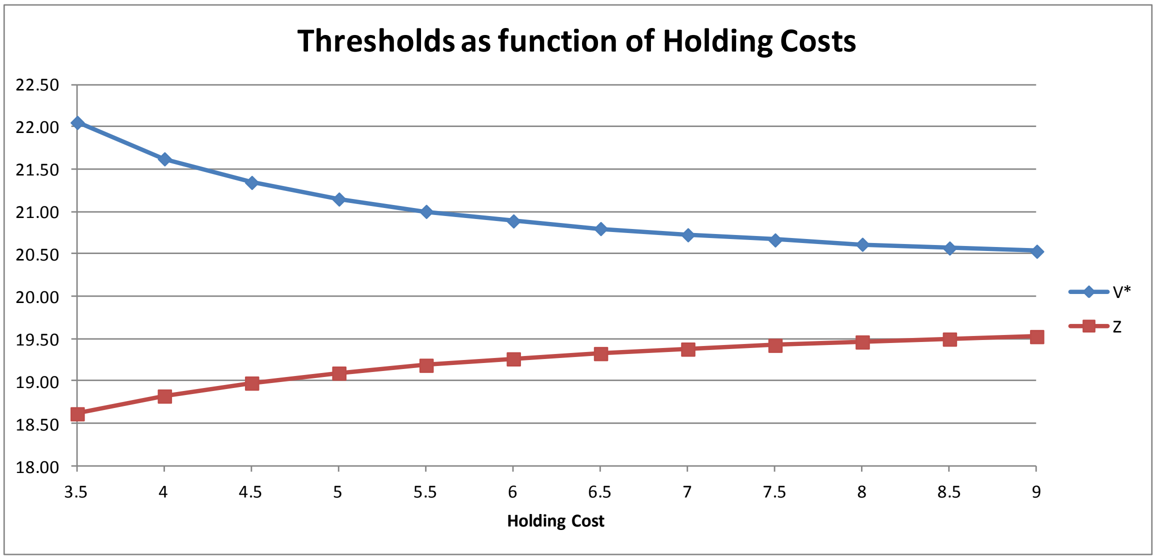


Since the holding cost influences the inferior and superior levels differently, the distance between them $\hat{V} - Z$ narrows as h increases. This effect is illustrated in Figure 2B using the base case values for the domain h>rX. The inferior and superior levels converge towards the exercise cost $X$ as the holding cost increases towards infinity. Although the profile for $\hat{V}$ is downward sloping while for $Z$ is upward sloping, their absolute gradients are greater for the lower values of h and less for the upper values.

Figure 2B




This feature may create a dilemma for the landowner who intends to sell the call option to extract the reserve. If the option seller sets a relatively high holding cost, the option value will be low. Although prospectors may want to buy a low priced option and speculate on a spot price rise, they will be encouraged to dispose of the option and walk away from the opportunity following a spot price fall because of the high holding cost. An alternative is for the option seller to set a relatively low holding cost. For this alternative, there exists the possibility that the holding cost is not sufficiently high and h<rX. Facing a low holding cost, prospectors will recognise the impossibility of exercising the option and will not purchase the option. There is a fine band of holding cost values that reassure the prospector that the purchased option will be eventually exercised rather than being allowed to lapse. The dilemma facing the option seller is to be able to identify this fine

band of holding cost values and then to select the most appropriate amongst the pairs of holding cost and option premium.

Since holding cost and volatility exert opposing pressures on the inferior and superior levels, a holding cost rise can to a certain extent moderate the effect on $Z$ and $\hat{V}$ arising from a volatility increase. When facing a volatility change, the option seller can therefore adjust the holding cost value in order to maintain a constant superior level. As the volatility approaches zero, and the stochastic model converts into the deterministic variant, $\lim_{\sigma \to 0} Z \to X, \lim_{\sigma \to 0} \hat{V} \to X$ since $\lim_{\sigma \to 0} \frac{\beta_{12}}{\beta_{12} - 1} \to 1$. The optimal extraction policy for the deterministic variant is both to purchase the option to extract and to exercise the option when the spot price is just sufficient to cover the per unit total cost.

When the spot price V falls to its inferior level $Z$, the extraction option has a zero value and is allowed to lapse. While $V < Z$, the option continues to have zero value and its erstwhile holder discontinues the holding cost payments. When the spot rate decline is reversed, the prospector should be able in principle to re-acquire the extraction option for a zero premium as soon as the spot price attains its inferior level.

3. Including the Convenience Yield

By assuming that the asset yield or convenience yield is proportional to the spot price, $\delta V$ with $\delta < r$, the risk neutral valuation relationship for the extraction option with a holding cost becomes:

$$\tfrac{1}{2}\sigma^2 V^2 \frac{\partial^2 C}{\partial V^2} + (r - \delta) V \frac{\partial C}{\partial V} - rC = h. \quad (10)$$

The option function takes the form:

$$C(V) = A_{31} V^{\beta_{21}} + A_{32} V^{\beta_{22}} - h/r. \quad (11)$$

where $\beta_{21},\beta_{22}=\left(\tfrac{1}{2}-\frac{r-\delta}{\sigma^2}\right)\pm\sqrt{\left(\tfrac{1}{2}-\frac{r-\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}$. For δ>0, $\beta_{21}>\beta_{11},\beta_{22}>\beta_{12}$. The option function solution is identical to (7) except for the change in coefficients.

Since the option has a zero value for $V=Z$ and the net operating revenue is just sufficient to reimburse the option for $V=\hat{V}$, then:

$$C(V)=\begin{cases}0 \text{ for } V\le Z,\\ A_{31}V^{\beta_{21}}+A_{32}V^{\beta_{22}}-\dfrac{h}{r} \text{ for } Z<V<\hat{V},\\ V-X \text{ for } V\ge\hat{V}.\end{cases} \tag{12}$$

The unknowns are derived from the value matching relationships and associated smooth pasting conditions. The inferior and superior levels, $Z$ and $\hat{V}$, do not have a closed-form solution and can only be found implicitly by solving the following simultaneous equations (derivation is similar to Appendix 4C (C.9 and C.10) with τ=0):

$$\hat{V}=\frac{\beta_{21}}{\beta_{21}-1}\left(X-\frac{h}{r}\left[1-\left(\frac{\hat{V}}{Z}\right)^{\beta_{22}}\right]\right)=\frac{-\beta_{22}}{1-\beta_{21}}\left(X-\frac{h}{r}\left[1-\left(\frac{\hat{V}}{Z}\right)^{\beta_{21}}\right]\right), \tag{13}$$

with:

$$A_{31}=\frac{-\beta_{22}}{(\beta_{21}-\beta_{22})Z^{\beta_{21}}}\frac{h}{r} \text{ and } A_{32}=\frac{\beta_{21}}{(\beta_{21}-\beta_{22})Z^{\beta_{22}}}\frac{h}{r}. \tag{C.5, C.6}$$

The combined option holding cost and convenience yield model is feasible for normal convenience yields and all positive holding costs. The insight from the combined model lies in the revelation that the option holding cost mediates between the poles of a real option formulation and a net present value appraisal, that is the real option solution for a zero value of the option holding cost and the NPV solution for an infinite holding cost. Although the effects of the holding cost and convenience yield on the solution are similar, the two variables cannot be interpreted as being substitutable. The convenience yield is determined exogenously and its value is a matter of empirical evaluation, whereas the option holding cost offers the option seller the potential leverage for regulating the exercise event and the prospector’s net profit at exercise.

4. Choice of Option Holding Cost and Royalty

When there exists a degree of flexibility over the level of the option holding cost and royalties and/or tax, the firm has to deliberate on the net revenues at exercise, as well as the value of the option holding cost itself. If the phenomenon under study involves a natural resource lease, the landowner may seek some freedom in structuring the deal and adjust the option holding cost. Government interests may be to optimise the total value of all revenue sources generated by the leasing deal. This implies that the lease option value, the option holding cost value and the production taxes and royalties should also be considered, recognising that the option holding cost determines the values of the other two quantities.

Both lessees and lessors should evaluate the economic consequences of alternative option holding costs and or their particular circumstances, and ascertain the most appropriate holding cost amount. As an illustration, suppose a government is the lessor and the natural resource under study has a known reserve volume of one unit. In making the holding cost decision, the government needs to consider the various sources of revenues generated by the lease contract stemming from the purchase price paid for the lease contract (the real option value), the stream of option holding cost payments, h, until extraction, and any royalty payment on the extracted reserve.

The royalty payment on the extracted reserve, which subsumes all payable taxes, is determined as a fraction $\tau$ of the underlying asset price at the extraction event. This means that the revenue apportioned to the lessee from extracting the reserve is $\hat{V}_h(1-\tau)$ instead of $\hat{V}_h$. The revised solution values for $Z_h$ and $\hat{V}_h$ are found from solving equations C.9 and C.10.

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The combined government value $V(V_h)$ accruing from the lease contract is the sum of the lease option purchase price, the present value of the option holding cost receipts and the present value of the royalty payment. If $V_0$ denotes the prevailing asset price at $t=0$ when the lease contract is originally negotiated, the purchase price of the lease contract is

$C_h(V_0)$. The present value of the accumulated holding cost receipts from $t = 0$ until the extraction event at $t = \bar{T}$ is the annuity value $h\left(1 - e^{-r\bar{T}}\right)/r$, where $\bar{T}$ is the anticipated time taken by the stochastic process $V$ to evolve from $V_0$ to $\hat{V}_h$ and is given by $\bar{T} = \dfrac{\ln\left(\hat{V}_h / V_0\right)}{\alpha - \frac{1}{2}\sigma^2}$ and $\alpha$ is the growth rate of the process $V$, with $\alpha > \frac{1}{2}\sigma^2$. Finally, at $t = 0$ the present value of the royalty payment available at extraction equals $\hat{V}_h \tau\, e^{-r\bar{T}}$. It follows that the combined value for $Z_h \le V_0 \le \hat{V}_h$ is:

$$V(V_h) = A_{31} V_0^{\beta_{21}} + A_{32} V_0^{\beta_{22}} + \left(\hat{V}_h \tau - \frac{h}{r}\right)\left(\frac{V_0}{\hat{V}_h}\right)^{\frac{r}{\alpha - \frac{1}{2}\sigma^2}}. \tag{14}$$

The expression $\dfrac{h}{r}\left(\dfrac{V_0}{\hat{V}_h}\right)^{\frac{r}{\alpha - \frac{1}{2}\sigma^2}}$ denotes the present value for the stream of holding cost payments foregone when the resource is extracted. When the prevailing asset price $V_0$ is equal to $\hat{V}_h$, the price triggering the extraction event, the lease agreement and extraction are coterminous. The lessee therefore does not make any option holding cost payments and $V(V_h)$ simplifies to $\hat{V}_h - X$.

For a specified royalty rate $\tau$, a government should strive to maximize as far as possible the combined value accruing from the lease contract by suitably adjusting the option holding cost. Since the combined value expression for $V(V_h)$ is not very tractable, the most convenient method for determining the optimal option holding cost is numerically. For each value, solve for $\hat{V}_h$ and $Z_h$ simultaneously from C.9 and C.10, and then proceed to evaluate $V(V_h)$ from (14) for a particular prevailing asset price $V_0$. The preferred option holding cost value occurs when the profile of the combined value $V(V_h)$ is maximized. For one set of h and $\tau$, the calculation for the base case data is shown in Figure 3B, which incidentally also allows for an escalating investment cost.

Figure 3B

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Improved Tourinho Model: Option to invest with holding cost & conyield** | | | | | | | |
| 2 | **Allows for X escalation and royalities** | | | | | | | |
| 3 | Oil price V | 60 | | | | | | |
| 4 | Extraction cost X | 20 | | | | | | |
| 5 | Volatility $\sigma$ | 0.20 | | | | | | |
| 6 | $\sigma$^2 | 0.04 | | | | | | |
| 7 | Risk-free rate | 10% | | | | | | |
| 8 | **Holding cost h** | **0.62** | | | | | | |
| 9 | Convenience yield | 2.0% | | | | | | |
| 10 | X Escalation Rate, g | 0.00 | | | | | | |
| 11 | Royalty, $\tau$ | **0.25** | | | | | | |
| 12 | V drift, $\alpha$ | 0.10 | | | | | | |
| 13 | T1 | -1.5000 | 0.5-(B7-B9-B10)/B6 | | | | | |
| 14 | T2 | 7.2500 | B13^2+2*(B7-B10)/B6 | | | | | |
| 15 | $\beta_1$ | 1.1926 | B13+SQRT(B14) | | | | | |
| 16 | $\beta_2$ | -4.1926 | B13-SQRT(B14) | | | | | |
| 17 | h/r | 6.2000 | B8/B7 | | | | | |
| 18 | u | -5.0000 | -2*B7/B6 | | | | | |
| 19 | V*(1-t) | 85.4606 | B20*(1-B11) | | | | | |
| 20 | V* | **113.9474** | | | | | | |
| 21 | Z | **11.8657** | | | | | | |
| 22 | A1 | 0.2526 | (-B16/(B15-B16))*B17/B21^B15 | | | | | |
| 23 | A2 | 43,828 | (-B15/B16)*B22*B21^(B15-B16) | | | | | |
| 24 | **C(V)** | **27.1505** | **B22*(B3^B15)+B23*(B3^B16)-B8/B7** | | | | | |
| 25 | V(V) EQ 14 | **43.3472** | **B22*(B3^B15)+B23*(B3^B16)+(B20*B11-B8/B7)*(B3/B20)^(B7/(B12-0.5*(B6)))** | | | | | |
| 26 | EQ C9 | 0.0000 | (-B16/(1-B16))*(B4-B17*(1-(B20/B21)^B15))-B19 | | | | | |
| 27 | EQ C10 | 0.0000 | (B15/(B15-1))*(B4-B17*(1-(B20/B21)^B16))-B19 | | | | | |
| 28 | SOLVER | 0.0000 | ABS(B26)+ABS(B27) | | | | | |
| 29 | SOLVER SET B28=0, CHANGING B20:B21 | | | | | | | |

By acting as a lever on the exercise event, the option holding cost represents an important policy instrument. Royalty payments on the extracted reserve act in the opposite direction to the option holding cost by prolonging the extraction event. The other model parameters are the extraction cost, the risk-free rate, the spot rate volatility and the convenience yield, which are all exogenous factors. Both parties to a concession or an unexploited resource need to be mindful of the costs and benefits arising from an option holding cost when structuring their extractive lease agreements and of the relationship between the holding cost level and the royalty rate.

## Appendix 4C: Holding Cost and Convenience Yield Model with Royalties

The expressions for $\hat{V}$ and $Z$ are derived for a positive royalty rate $\tau$. The value matching relationships for the two levels are respectively:

$$A_{31}Z^{\beta_{21}} + A_{32}Z^{\beta_{22}} - \frac{h}{r} = 0, \qquad \text{C.1}$$

$$A_{31}\hat{V}^{\beta_{21}} + A_{32}\hat{V}^{\beta_{22}} - \frac{h}{r} = \hat{V}\left(1-\tau\right) - X. \qquad \text{C.2}$$

The two smooth pasting conditions associated with C.1 and C.2 are:

$$A_{31}\beta_{21}Z^{\beta_{21}-1} + A_{32}\beta_{22}Z^{\beta_{22}-1} = 0, \qquad \text{C.3}$$

$$A_{31}\beta_{21}\hat{V}^{\beta_{21}-1} + A_{32}\beta_{22}\hat{V}^{\beta_{22}-1} = 1-\tau. \qquad \text{C.4}$$

Using C.1 and C.2 to solve for $A_{31}$ and $A_{32}$ yields:

$$A_{31} = \frac{-\beta_{22}}{\beta_{21}-\beta_{22}}\frac{h}{r}\frac{1}{Z^{\beta_{21}}}, \qquad \text{C.5}$$

$$A_{32} = \frac{\beta_{21}}{\beta_{21}-\beta_{22}}\frac{h}{r}\frac{1}{Z^{\beta_{22}}}. \qquad \text{C.6}$$

Using C.2 and C.4 to solve for $A_{31}$ and $A_{32}$ yields:

$$A_{31} = \frac{\hat{V}\left(1-\tau\right) - \beta_{22}\left(\hat{V}\left(1-\tau\right) + h/r - X\right)}{\left(\beta_{21}-\beta_{22}\right)\hat{V}^{\beta_{21}}}, \qquad \text{C.7}$$

$$A_{32} = \frac{\beta_{21}\left(\hat{V}\left(1-\tau\right) + h/r - X\right) - \hat{V}\left(1-\tau\right)}{\left(\beta_{21}-\beta_{22}\right)\hat{V}^{\beta_{22}}}. \qquad \text{C.8}$$

Eliminating $A_{31}$ from C.5 and C.7, and $A_{32}$ from C.6 and C.8 yields:

$$\hat{V}\left(1-\tau\right) = \frac{-\beta_{22}}{1-\beta_{22}}\left(X - \frac{h}{r}\left[1-\left(\frac{\hat{V}}{Z}\right)^{\beta_{21}}\right]\right), \qquad \text{C.9}$$

$$\hat{V}\left(1-\tau\right) = \frac{\beta_{21}}{\beta_{21}-1}\left(X - \frac{h}{r}\left[1-\left(\frac{\hat{V}}{Z}\right)^{\beta_{22}}\right]\right). \qquad \text{C.10}$$

Solutions for $\hat{V}$ and $Z$ can be found from C.9 and C.10.

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[1] The Tourinho (1979b) thesis was not published, but the working paper (1979a) based on his research is available from the University of California, Berkeley, Institute of Business and Economic Research.